

A Posteriori Error Bounds for the Empirical Interpolation Method

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Abstract

We present rigorous *a posteriori* error bounds for the Empirical Interpolation Method (EIM). The essential ingredients are (i) analytical upper bounds for the parametric derivatives of the function to be approximated, (ii) the EIM “Lebesgue constant,” and (iii) information concerning the EIM approximation error at a finite set of points in parameter space. The bound is computed “offline” and is valid over the entire parameter domain; it is thus readily employed in (say) the “online” reduced basis context. We present numerical results that confirm the validity of our approach.

Résumé

Un estimateur a posteriori d’erreur pour la méthode d’interpolation empirique. On introduit des bornes d’erreur *a posteriori* rigoureuses pour la méthode d’interpolation empirique, EIM en abrégé (pour Empirical Interpolation Method). Les ingrédients essentiels sont (i) des bornes analytiques des dérivées par rapport au paramètre de la fonction à interpoler, (ii) une “constante de Lebesgue” de EIM, et (iii) de l’information sur l’erreur d’approximation commise par EIM en un nombre fini de points dans l’espace des paramètres. La borne, une fois pré-calculée “hors-ligne”, est valable sur tout l’espace des paramètres; elle peut donc être utilisée directement telle quelle dans les applications (étape “en ligne” des calculs dans le contexte de la méthode des bases réduites). On montre des résultats numériques qui confirment la validité de notre approche.

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Soit $\mathcal{F}_M(\cdot; \mu) \equiv \sum_{k=1}^M \phi_M(\mu) \mathcal{F}(\cdot; \mu_m)$ une approximation de $\mathcal{F} \in C^\infty(\mathcal{D}, L^\infty(\Omega))$, fonction qui joue le rôle d'un coefficient paramétré “non-affine” dans la méthode des bases réduites. La méthode d'interpolation empirique (EIM) sert à construire $\mathcal{F}_M(\cdot; \mu)$; elle fournit aussi un estimateur de l'erreur d'interpolation $e_M(\mu) \equiv \|\mathcal{F}(\cdot; \mu) - \mathcal{F}_M(\cdot; \mu)\|_{L^\infty(\Omega)}$, qui n'est pas rigoureux mais qui est souvent suffisamment précis. Dans ce travail, nous construisons rigoureusement une borne supérieure d'erreur *a posteriori* pour $e_M(\mu)$.

D'abord, nous rappelons ce qu'est la méthode EIM. Ses ingrédients essentiels sont (i) la construction d'un espace d'approximation $W_M \equiv \text{span}\{\mathcal{F}(\cdot; \mu_m)\}_{m=1}^M$ avec quelques valeurs μ_m , $1 \leq m \leq M$, sélectionnées par un algorithme glouton, pour le paramètre $\mu \in \mathcal{D}$, et (ii) la sélection d'un ensemble de nœuds d'interpolation $T_M \equiv \{t_1 \in \Omega, \dots, t_M \in \Omega\}$ associé à W_M . L'approximation $\mathcal{F}_M(\cdot; \mu) \in W_M$ est définie comme l'interpolant de $\mathcal{F}(\cdot; \mu)$ sur l'ensemble T_M .

Ensuite, nous introduisons notre nouvelle borne d'erreur. Pour cela, nous développons $\mathcal{F}(x; \mu)$ en une série de Taylor à plusieurs variables, avec un ensemble fini Φ de points dans l'espace des paramètres $\mathcal{D} \subset \mathbb{R}^P$. Pour tout entier positif I , nous supposons $\max_{\mu \in \mathcal{D}} \max_{\beta \in \mathcal{M}_I^P} \|\mathcal{F}^{(\beta)}(\cdot; \mu)\|_{L^\infty(\Omega)} \leq \sigma_I (< \infty)$, avec \mathcal{M}_I^P l'ensemble de tous les multi-indices positifs $\beta \equiv (\beta_1, \dots, \beta_P)$ de dimension P et de longueur $\sum_{i=1}^P \beta_i = I$ (pour $1 \leq i \leq P$, β_i est un entier positif) et $\mathcal{F}^{(\beta)}$ la dérivée β -ième de \mathcal{F} par rapport à μ . Puis nous posons $\rho_\Phi \equiv \max_{\mu \in \mathcal{D}} \min_{\tau \in \Phi} |\mu - \tau|$, nous définissons une “constante de Lebesgue” $\Lambda_M \equiv \sup_{x \in \Omega} \sum_{m=1}^M |V_m^M(x)|$, où $V_m^M(x) \in W_M$ sont les fonctions caractéristiques $V_m^M(t_n) \equiv \delta_{mn}$, $1 \leq m, n \leq M$, et nous pouvons alors prouver notre Proposition 2.1, soit : $\max_{\mu \in \mathcal{D}} e_M(\mu) \leq \delta_{M,p}$, avec une borne $\delta_{M,p}$ définie en (2).

Enfin, nous présentons des résultats numériques avec une fonction gaussienne $\mathcal{F}(x; \mu) = \exp(-((x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2)/2\alpha^2)$ sur $\Omega \equiv (0, 1)^2$. Avec un seul paramètre (scalaire) $\alpha \equiv \mu \in \mathcal{D}_I \equiv [0.1, 1]$ et $(\bar{x}_1, \bar{x}_2) = (0.5, 0.5)$ fixé, on calcule $\max_{\mu \in \Xi_{\text{train}}} e_M(\mu)$ et $\delta_{M,p}$, $p = 1, 2, 3, 4$ pour $1 \leq M \leq M_{\text{max}}$ (Fig. 1). Nous observons que les bornes d'erreur commencent par décroître puis atteignent un “plateau” en M . On calcule aussi Λ_M et l'effectivité moyenne $\bar{\eta}_{M,p}$ pour $p = 4$ en tant que fonctions de M : la constante de Lebesgue ne croît que légèrement avec M , et les bornes d'erreurs sont très précises pour de petites valeurs de M . Avec deux paramètres $(\bar{x}_1, \bar{x}_2) \equiv \mu \in \mathcal{D}_{\text{II}} \equiv [0.4, 0.6]^2$ et $\alpha = 0.1$ fixé, les résultats sont similaires au cas d'un seul paramètre (Fig. 2).

1. Introduction

The Empirical Interpolation Method (EIM), introduced in [1], serves to construct “affine” approximations of “non-affine” parametrized functions. The method is frequently applied in reduced basis approximation of parametrized partial differential equations with non-affine parameter dependence [4]; the affine approximation of the coefficient functions is crucial for computational efficiency. In previous work [1,4] an estimator for the interpolation error is developed; this estimator is often very accurate, however it is not a rigorous upper bound. In this paper, we develop a rigorous *a posteriori* upper bound for the interpolation error and we present numerical results that confirm the validity of our approach.

To begin, we summarize the EIM [1,4]. We are given a function $\mathcal{G} : \Omega \times \mathcal{D} \rightarrow \mathbb{R}$ such that, for all $\mu \in \mathcal{D}$, $\mathcal{G}(\cdot; \mu) \in L^\infty(\Omega)$; here, $\mathcal{D} \subset \mathbb{R}^P$ is the parameter domain, $\Omega \subset \mathbb{R}^2$ is the spatial domain—a point in which shall be denoted by $x = (x_1, x_2)$ —and $L^\infty(\Omega) \equiv \{v \mid \text{ess sup}_{v \in \Omega} |v(x)| < \infty\}$. We introduce a finite train sample $\Xi_{\text{train}} \subset \mathcal{D}$ which shall serve as our \mathcal{D} surrogate, and a triangulation $\mathcal{T}_N(\Omega)$ of Ω with N vertices over which we shall in practice realize $\mathcal{G}(\cdot; \mu)$ as a piecewise linear function.

We first define the nested EIM approximation spaces $W_M^{\mathcal{G}}$, $1 \leq M \leq M_{\text{max}}$. We first choose $\mu_1 \in \mathcal{D}$,

compute $g_1 \equiv \mathcal{G}(\cdot; \mu_1)$, and define $W_1^{\mathcal{G}} \equiv \text{span}\{g_1\}$; then, for $2 \leq M \leq M_{\max}$, we determine $\mu_M \equiv \arg \max_{\mu \in \Xi_{\text{train}}} \inf_{z \in W_{M-1}^{\mathcal{G}}} \|\mathcal{G}(\cdot; \mu) - z\|_{L^\infty(\Omega)}$, compute $g_M \equiv \mathcal{G}(\cdot; \mu_M)$, and define $W_M^{\mathcal{G}} \equiv \text{span}\{g_m\}_{m=1}^M$.

We next introduce the nested set of EIM interpolation nodes $T_M^{\mathcal{G}} \equiv \{t_1, \dots, t_M\}$, $1 \leq M \leq M_{\max}$. We first set $t_1 \equiv \arg \sup_{x \in \Omega} |g_1(x)|$ and $q_1 \equiv g_1/g_1(t_1)$; then, for $2 \leq M \leq M_{\max}$, we solve the linear system $\sum_{j=1}^{M-1} \omega_j^{M-1} q_j(t_i) = g_M(t_i)$, $1 \leq i \leq M-1$, and set $r_M(x) = g_M(x) - \sum_{j=1}^{M-1} \omega_j^{M-1} q_j(x)$, $t_M \equiv \arg \sup_{x \in \Omega} |r_M(x)|$, and $q_M = r_M/r_M(t_M)$. For $1 \leq M \leq M_{\max}$, we define the matrix $B^M \in \mathbb{R}^{M \times M}$ such that $B_{ij}^M \equiv q_j(t_i)$, $1 \leq i, j \leq M$; we note that B^M is lower triangular with unity diagonal and that $\{q_m\}_{m=1}^M$ is a basis for $W_M^{\mathcal{G}}$ [1,4].

We are now given a function $\mathcal{H} : \Omega \times \mathcal{D} \rightarrow \mathbb{R}$ such that, for all $\mu \in \mathcal{D}$, $\mathcal{H}(\cdot; \mu) \in L^\infty(\Omega)$. We define for any $\mu \in \mathcal{D}$ the EIM interpolant $\mathcal{H}_{W_M^{\mathcal{G}}}(\cdot; \mu) \in W_M^{\mathcal{G}}$ as the interpolant of $\mathcal{H}(\cdot; \mu)$ over the set $T_M^{\mathcal{G}}$. Specifically $\mathcal{H}_{W_M^{\mathcal{G}}}(\cdot; \mu) \equiv \sum_{m=1}^M \phi_M m(\mu) q_m$, where $\sum_{j=1}^M B_{ij}^M \phi_M j(\mu) = \mathcal{H}(t_i; \mu)$, $1 \leq i \leq M$. Note that the determination of the coefficients $\phi_M m(\mu)$ requires only $\mathcal{O}(M^2)$ computational cost.

Finally, we define a ‘‘Lebesgue constant’’ [6] $\Lambda_M \equiv \sup_{x \in \Omega} \sum_{m=1}^M |V_m^M(x)|$, where $V_m^M \in W_M^{\mathcal{G}}$ are the characteristic functions of $W_M^{\mathcal{G}}$ satisfying $V_m^M(t_n) \equiv \delta_{mn}$, $1 \leq m, n \leq M$; here, δ_{mn} is the Kronecker delta symbol. We recall that (i) the set of all characteristic functions $\{V_m^M\}_{m=1}^M$ is a basis for $W_M^{\mathcal{G}}$, and (ii) the Lebesgue constant Λ_M satisfies $\Lambda_M \leq 2^M - 1$ [1,4]. In applications, the actual asymptotic behavior of Λ_M is much better, as we shall observe subsequently.

2. A Posteriori Error Estimation

We now develop the new and rigorous upper bound for the error associated with the empirical interpolation of a function $\mathcal{F} : \Omega \times \mathcal{D} \rightarrow \mathbb{R}$. We shall assume that \mathcal{F} is parametrically smooth; for simplicity here, we suppose $\mathcal{F} \in C^\infty(\mathcal{D}, L^\infty(\Omega))$. Our bound depends on the parametric derivatives of \mathcal{F} and on the EIM interpolant of these derivatives. For this reason, we introduce a multi-index of dimension P , $\beta \equiv (\beta_1, \dots, \beta_P)$, where the β_i , $1 \leq i \leq P$, are non-negative integers; we further define the length $|\beta| \equiv \sum_{i=1}^P \beta_i$, and denote the set of all distinct multi-indices β of dimension P of length I by \mathcal{M}_I^P . The cardinality of \mathcal{M}_I^P is given by $\text{card}(\mathcal{M}_I^P) = \binom{P+I-1}{I}$. For any multi-index β , we define

$$\mathcal{F}^{(\beta)}(x; \mu) \equiv \frac{\partial^{|\beta|} \mathcal{F}}{\partial \mu_{(1)}^{\beta_1} \dots \partial \mu_{(P)}^{\beta_P}}(x; \mu); \quad (1)$$

we require that $\max_{\mu \in \mathcal{D}} \max_{\beta \in \mathcal{M}_I^P} \|\mathcal{F}^{(\beta)}(\cdot; \mu)\|_{L^\infty(\Omega)} \leq \sigma_p (< \infty)$ for non-negative integer p .

Given any $\mu \in \mathcal{D}$, we define for $1 \leq M \leq M_{\max}$ the interpolants of $\mathcal{F}(\cdot; \mu)$ and $\mathcal{F}^{(\beta)}(\cdot; \mu)$ as $\mathcal{F}_M(\cdot; \mu) \equiv \mathcal{F}_{W_M^{\mathcal{F}}}(\cdot; \mu)$ and $(\mathcal{F}^{(\beta)})_M(\cdot; \mu) \equiv \mathcal{F}_{W_M^{\mathcal{F}}}^{(\beta)}(\cdot; \mu)$, respectively. We emphasize that both interpolants $\mathcal{F}_M(\cdot; \mu)$ and $(\mathcal{F}^{(\beta)})_M(\cdot; \mu)$ lie in the same space $W_M^{\mathcal{F}}$ —we do not introduce a separate space, $W_M^{\mathcal{F}^{(\beta)}}$, spanned by solutions of $\mathcal{F}^{(\beta)}(\cdot; \mu_M)$, $1 \leq M \leq M_{\max}$. It is thus readily demonstrated that $(\mathcal{F}^{(\beta)})_M(\cdot; \mu) = (\mathcal{F}_M)^{(\beta)}(\cdot; \mu)$, which we thus henceforth denote $\mathcal{F}_M^{(\beta)}(\cdot; \mu)$.¹ Note that $\mathcal{F}_M^{(\beta)}(\cdot; \mu) \in W_M^{\mathcal{F}}$ is the unique interpolant satisfying $\mathcal{F}_M^{(\beta)}(t_m; \mu) = \mathcal{F}^{(\beta)}(t_m; \mu)$, $1 \leq m \leq M$. We can further demonstrate [3] in certain cases that if $\mathcal{F}_M(\cdot; \mu)$ tends to $\mathcal{F}(\cdot; \mu)$ as $M \rightarrow \infty$ then $\mathcal{F}_M^{(\beta)}(\cdot; \mu)$ tends to $\mathcal{F}^{(\beta)}(\cdot; \mu)$ as $M \rightarrow \infty$.

¹ Let $Z_q = [q_1 \dots q_M]$ and $\bar{t}_M = [t_1 \dots t_M]$. We then have $\mathcal{F}_M(\cdot; \mu) = Z_q(B^M)^{-1} \mathcal{F}(\bar{t}_M; \mu)$ and $(\mathcal{F}^{(\beta)})_M(\cdot; \mu) = Z_q(B^M)^{-1} \mathcal{F}^{(\beta)}(\bar{t}_M; \mu)$. Since B^M and the basis functions q_i , $1 \leq i \leq M$, are independent of μ , it follows that $(\mathcal{F}_M)^{(\beta)}(\cdot; \mu) = (Z_q(B^M)^{-1} \mathcal{F}(\bar{t}_M; \mu))^{(\beta)} = Z_q(B^M)^{-1} \mathcal{F}^{(\beta)}(\bar{t}_M; \mu) = (\mathcal{F}^{(\beta)})_M(\cdot; \mu)$.

We now develop the interpolation error upper bound. To begin, we introduce a set of points $\Phi \subset \mathcal{D}$ of size n_Φ and define $\rho_\Phi \equiv \max_{\mu \in \mathcal{D}} \min_{\tau \in \Phi} |\mu - \tau|$; here $|\cdot|$ is the usual Euclidean norm. We then define

$$\delta_{M,p} \equiv (1 + \Lambda_M) \frac{\sigma_p}{p!} \rho_\Phi^p P^{p/2} + \sup_{\tau \in \Phi} \left(\sum_{j=0}^{p-1} \frac{\rho_\Phi^j}{j!} P^{j/2} \max_{\beta \in \mathcal{M}_j^P} \|\mathcal{F}^{(\beta)}(\cdot; \tau) - \mathcal{F}_M^{(\beta)}(\cdot; \tau)\|_{L^\infty(\Omega)} \right). \quad (2)$$

We can now demonstrate

Proposition 2.1 *For given positive integer p , $\max_{\mu \in \mathcal{D}} \|\mathcal{F}(\cdot; \mu) - \mathcal{F}_M(\cdot; \mu)\|_{L^\infty(\Omega)} \leq \delta_{M,p}$, $\forall \mu \in \mathcal{D}$, $1 \leq M \leq M_{\max}$.*

PROOF. We present the proof for $P = 1$ and refer the reader to [3] for the general case $P \geq 1$. For brevity, we first define (assuming existence) $\mathcal{A}_\mathcal{G}^p(\tau, \mu) \equiv \sum_{j=0}^{p-1} \mathcal{G}^{(j)}(\cdot; \tau) \frac{(\mu - \tau)^j}{j!}$ as the first p terms in the Taylor series of \mathcal{G} around τ . We then choose τ as $\tau^*(\mu) \equiv \arg \min_{\bar{\tau} \in \Phi} |\mu - \bar{\tau}|$. We note that

$$\|\mathcal{F}(\cdot; \mu) - \mathcal{F}_M(\cdot; \mu)\|_{L^\infty(\Omega)} \leq \|\mathcal{F}(\cdot; \mu) - \mathcal{A}_\mathcal{F}^p(\tau^*, \mu)\|_{L^\infty(\Omega)} + \|\mathcal{A}_\mathcal{F}^p(\tau^*, \mu) - \mathcal{F}_M(\cdot; \mu)\|_{L^\infty(\Omega)} \quad (3)$$

for all $\mu \in \mathcal{D}$. We recall the univariate Taylor series expansion with remainder in integral form $\mathcal{F}(x; \mu) = \mathcal{A}_\mathcal{F}^p(\tau, \mu) + \int_\tau^\mu \mathcal{F}^{(p)}(x; \bar{\tau}) \frac{(\mu - \bar{\tau})^{p-1}}{(p-1)!} d\bar{\tau}$. We can now bound the first term on the right hand side of (3) by

$$\|\mathcal{F}(\cdot; \mu) - \mathcal{A}_\mathcal{F}^p(\tau^*, \mu)\|_{L^\infty(\Omega)} \leq \left\| \int_{\tau^*}^\mu \left\| \mathcal{F}^{(p)}(\cdot; \bar{\tau}) \frac{(\mu - \bar{\tau})^{p-1}}{(p-1)!} \right\|_{L^\infty(\Omega)} d\bar{\tau} \right\| \leq \frac{\sigma_p}{p!} \rho_\Phi^p \quad (4)$$

for all $\mu \in \mathcal{D}$. For the second term in (3), we obtain

$$\|\mathcal{A}_\mathcal{F}^p(\tau^*, \mu) - \mathcal{F}_M(\cdot; \mu)\|_{L^\infty(\Omega)} \leq \|\mathcal{A}_\mathcal{F}^p(\tau^*, \mu) - \mathcal{A}_{\mathcal{F}_M}^p(\tau^*, \mu)\|_{L^\infty(\Omega)} + \|\mathcal{A}_{\mathcal{F}_M}^p(\tau^*, \mu) - \mathcal{F}_M(\cdot; \mu)\|_{L^\infty(\Omega)} \quad (5)$$

for all $\mu \in \mathcal{D}$. For the first term in (5) we note that

$$\|\mathcal{A}_\mathcal{F}^p(\tau^*, \mu) - \mathcal{A}_{\mathcal{F}_M}^p(\tau^*, \mu)\|_{L^\infty(\Omega)} \leq \sup_{\tau \in \Phi} \left(\sum_{j=0}^{p-1} \frac{\rho_\Phi^j}{j!} \left\| \mathcal{F}^{(j)}(\cdot; \tau) - \mathcal{F}_M^{(j)}(\cdot; \tau) \right\|_{L^\infty(\Omega)} \right), \quad \forall \mu \in \mathcal{D}. \quad (6)$$

From the definition of the characteristic functions V_m^M , we obtain $\sum_{j=0}^{p-1} \mathcal{F}_M^{(j)}(x; \tau^*) \frac{(\mu - \tau^*)^j}{j!} - \mathcal{F}_M(x; \mu) = \sum_{m=1}^M \left[\sum_{j=0}^{p-1} \mathcal{F}_M^{(j)}(t_m; \tau^*) \frac{(\mu - \tau^*)^j}{j!} - \mathcal{F}_M(t_m; \mu) \right] V_m^M(x)$. We then invoke the interpolation property (for any non-negative integer j) $\mathcal{F}_M^{(j)}(t_m; \mu) = \mathcal{F}^{(j)}(t_m; \mu)$, $1 \leq m \leq M$, and the definition of the Lebesgue constant Λ_M , to bound the second term in (5) by

$$\|\mathcal{A}_{\mathcal{F}_M}^p(\tau^*, \mu) - \mathcal{F}_M(\cdot; \mu)\|_{L^\infty(\Omega)} \leq \|\mathcal{A}_\mathcal{F}^p(\tau^*, \mu) - \mathcal{F}(\cdot; \mu)\|_{L^\infty(\Omega)} \Lambda_M \leq \frac{\sigma_p}{p!} \rho_\Phi^p \Lambda_M, \quad \forall \mu \in \mathcal{D}. \quad (7)$$

The desired result (for $P = 1$) directly follows.

We make several remarks concerning this result. First, we may choose p such that the two terms in (2) balance—a higher p will reduce the contribution of the first term but will increase the contribution of the second term. Second, we note that the bound $\delta_{M,p}$ is μ -independent. We can readily develop a μ -dependent bound by replacing ρ_Φ with the actual distance between μ and the closest $\tau \in \Phi$; this μ -dependent bound can serve (i) to adaptively construct an economical point set Φ , and (ii) to replace the true (expensive) error in the greedy identification of the EIM spaces $W_M^\mathcal{G}$. Third, we can increase the sharpness of the bound by localizing the derivative bounds σ_p : this is best achieved through an “ hp ” approach for the EIM; we note that the “ hp ” framework developed in [2] for the reduced basis method

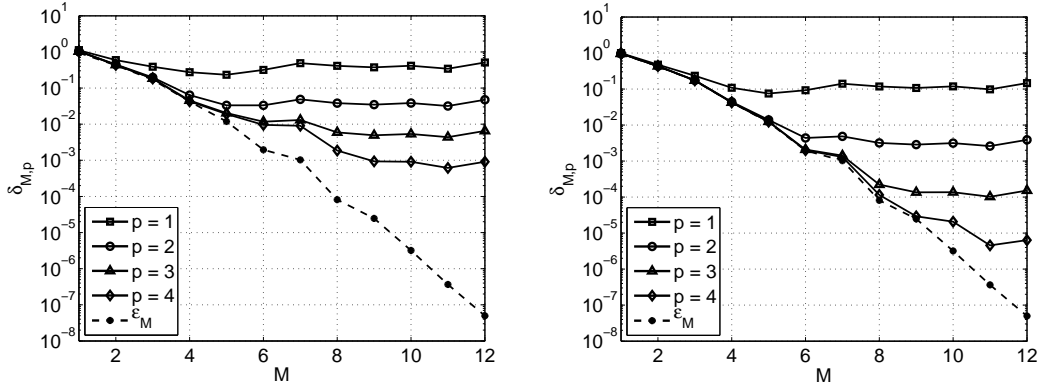


Figure 1. Error bounds $\delta_{M,p}$ for $P = 1$ and $p = 1, 2, 3, 4$ with $n_\Phi = 41$ (left) and $n_\Phi = 141$ (right).

readily adapts to the EIM (see also [5] for an alternative approach). Fourth, we note that in the “limit” $\rho_\Phi \rightarrow 0$ the effectivity of the bound approaches unity; of course, we will never in practice let $\rho_\Phi \rightarrow 0$ because this implies the computation of the interpolant at every point in \mathcal{D} . Fifth, we note that our bound at no point requires computation of spatial derivatives of the function to be approximated.

We conclude this section by summarizing the computational cost associated with $\delta_{M,p}$. We assume that the bounds σ_p can be obtained analytically. We compute Λ_M in $\mathcal{O}(M^2\mathcal{N})$ operations, and we compute the interpolation errors $\|\mathcal{F}^{(\beta)}(\cdot; \tau) - \mathcal{F}_M^{(\beta)}(\cdot; \tau)\|_{L^\infty(\Omega)}$, $0 < |\beta| < p - 1$, for all $\tau \in \Phi$, in $\mathcal{O}(n_\Phi M \mathcal{N}) \sum_{j=0}^{p-1} \text{card}(\mathcal{M}_j^P)$ operations (we assume $M \ll \mathcal{N}$); certainly the growth of \mathcal{M}_p^P will preclude large P . Note the computational cost is “offline” only—the bound is valid for all $\mu \in \mathcal{D}$.

3. Numerical Results

We shall consider the empirical interpolation of a Gaussian function $\mathcal{F}(\cdot; \mu)$ over two different parameter domains $\mathcal{D} = \mathcal{D}_I$ and $\mathcal{D} = \mathcal{D}_{II}$. The spatial domain is $\Omega \equiv [0, 1]^2$; we introduce a triangulation $\mathcal{T}_\mathcal{N}(\Omega)$ with $\mathcal{N} = 2601$ vertices. We shall compare our bound with the true interpolation error over the parameter domain. To this end, we define the maximum error $\varepsilon_M \equiv \max_{\mu \in \Xi_{\text{train}}} e_M(\mu)$ and the average effectivity $\bar{\eta}_{M,p} \equiv \text{mean}_{\mu \in \Xi_{\text{test}}} \delta_{M,p}/e_M(\mu)$; here, $e_M(\mu) \equiv \|\mathcal{F}(\cdot; \mu) - \mathcal{F}_M(\cdot; \mu)\|_{L^\infty(\Omega)}$, and $\Xi_{\text{test}} \subset \mathcal{D}$ is a test sample of finite size $n_{\Xi_{\text{test}}}$.

We first consider the case $\mathcal{D} = \mathcal{D}_I \equiv [0.1, 1]$ and hence $P = 1$; we let $\mathcal{F}(x; \mu) = \mathcal{F}_I(x; \mu) \equiv \exp(-((x_1 - 0.5)^2 + (x_2 - 0.5)^2)/2\mu^2)$. We introduce an equidistant train sample $\Xi_{\text{train}} \subset \mathcal{D}$ of size 500; we take $\mu_1 = 1$ and pursue the EIM with $M_{\text{max}} = 12$. In Figure 1 we report ε_M and $\delta_{M,p}$, $p = 1, 2, 3, 4$, for $1 \leq M \leq M_{\text{max}}$; we consider $n_\Phi = 41$ and $n_\Phi = 141$ ($\rho_\Phi = 1.125 \text{E-}2$ and $\rho_\Phi \approx 3.21 \text{E-}3$, respectively). We observe that the error bounds initially decrease, but then “plateau” in M . The bounds are very sharp for sufficiently small M , but eventually the first term in (2) dominates and compromises the sharpness of the bounds; for larger p , the bound is better for a larger range of M . We find that $1 \leq \Lambda_M \leq 5.18$ for $1 \leq M \leq M_{\text{max}}$ and, for the case $p = 4$ with $n_\Phi = 141$, $\bar{\eta}_{M,p} \sim \mathcal{O}(10)$ ($n_{\Xi_{\text{test}}} = 150$) except for large M . The modest growth of the Lebesgue constant is crucial to the good effectivity.

We next consider the case $\mathcal{D} = \mathcal{D}_{II} \equiv [0.4, 0.6]^2$ and hence $P = 2$; we introduce $\mathcal{F} = \mathcal{F}_{II}(x; \mu) = \exp(-((x_1 - \mu_{(1)})^2 + (x_2 - \mu_{(2)})^2)/2(0.1)^2)$, where $\mu \equiv (\mu_{(1)}, \mu_{(2)})$. We introduce a deterministic grid $\Xi_{\text{train}} \subset \mathcal{D}$ of size 1600; we take $\mu_1 = (0.4, 0.4)$ and pursue the EIM with $M_{\text{max}} = 60$. In Figure 2 we report ε_M and $\delta_{M,p}$, $p = 1, 2, 3, 4$, for $1 \leq M \leq M_{\text{max}}$; we consider $n_\Phi = 100$ and $n_\Phi = 1600$ ($\rho_\Phi \approx 1.57 \text{E-}2$ and

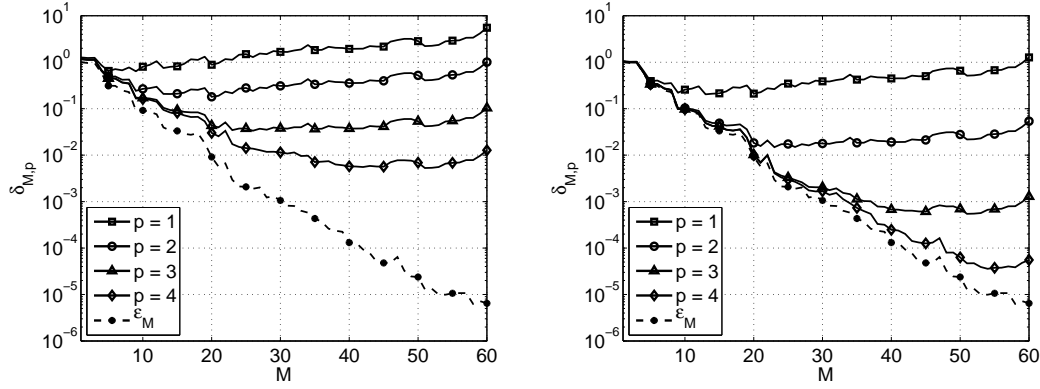


Figure 2. Error bounds $\delta_{M,p}$ for $P = 2$ and $p = 1, 2, 3, 4$ with $n_\Phi = 100$ (left) and $n_\Phi = 1600$ (right).

3.63E-3, respectively). We observe the same behavior as for the $P = 1$ case: the errors initially decrease, but then “plateau” in M depending on the particular value of p . We find that $1 \leq \Lambda_M \leq 39.9$ and, for the case $p = 4$ with $n_\Phi = 1600$, $\bar{\eta}_{M,p} \sim \mathcal{O}(10)$ ($n_{\Xi_{\text{test}}} = 225$) for $1 \leq M \leq M_{\text{max}}$.

Our results demonstrate that we can gainfully increase p —the number of terms in the Taylor series expansion—in order to reduce the role of the first term of $\delta_{M,p}$ and to limit the size of Φ . We also note that for the examples presented here the terms in the sum of (2) are well behaved, even though (for our $P = 2$ example in particular) it is not obvious that the space $W_M^{\mathcal{F}}$ contains good interpolants of the functions $\mathcal{F}^{(\beta)}(\cdot, \mu)$, $|\beta| \neq 0$.

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